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A SOLUTION OF THE TWO-DIMENSIONAL TURBULENT VISCOUS CURVED JET USING THE IBM 7090 COMPUTER

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A SOLUTION OF THE TWO-DIMENSIONAL TURBULENT VISCOUS CURVED JET USING THE IBM 7090 COMPUTER

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FOREWORD

The study presented in this report covers a phase of a theoretical investigation of curved jets supported by the U.S. Army transportation Corps during the year 1960-1961 through the Air Programs Office of the Office of Naval Research, under the terms of Contract Nonr-2747(00).

SUMMARY

A numerical solution of the two-dimensional curved turbulent and incompressible jet flow, using the IBM 7090 computer, is presented. This solution is a straight-forward extension of the classical straight jet solution. Its only limitation is that it assumes similar velocity profiles.

The solution is obtained by reducing the Navier Stokes partial differential equations of the curved jet flow to a third-order total differential equation of the Hartree-Skan type. This equation is integrated numerically on the IBM 7090 computer using a Runge-Kutta subroutine. The boundary conditions are carefully established and discussed. The complete program for the numerical solution is included in the report. A duplicate IBM deck can be obtained from Aerophysics Company upon request. Typical numerical results, in the form of velocity and pressure distribution across the jet are presented and discussed.

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LIST OF SYMBOLS

A	Constant, shape parameter of the curved jet
b	width of jet
C	constant $= \frac{A\sigma}{2}$
$F(\eta)$	unknown function of the problem
$G(\eta)$	$= F'(\eta)$
$H(\eta)$	$= G'(\eta)$
K	jet curvature
m	mass flow
p	static pressure
R	radius of curvature of jet
s	part I: fixed characteristic length from the orifice, counted along the x-axis
	part II: curvilinear distance: $ds^2 = dx^2 + dy^2$
U(x)	maximum jet velocity
u	jet velocity component in x-direction
v	jet velocity component in y-direction
x, y	curvilinear system of coordinates, as defined in the text and on figure 1
α	constant
β	constant
ϵ	virtual kinematic viscosity
η	$= \sigma\, y/x$, dimensionless variable
μ	viscosity
ρ	density
σ	free constant, assumed equal to 8. (This is arbitrary, represents only an order of magnitude)
χ_1	empirical constant
Ψ	stream function

<u>Subscript</u>

s	conditions at the distance s from the orifice

INTRODUCTION

In spite of the remarkable success of inviscid theories to predict the performance of hovering ground effect machines, it cannot be concluded that viscous phenomena do not sometimes play an important role in the understanding of the flow mechanics of hovering annular jets in ground effect. In particular, the lack of an analytical grasp of viscous effects has prevented until now the establishment of a satisfactory mathematical theory of the stability of ground effect machines.

The major viscous effect associated with annular jet flow concerns entrainment of air from the central cavity, the so-called "air cushion", by the peripheral jet. When the base of the machine is parallel to the ground, this entrainment results in a quasi-rotational flow under the machine, which can be ignored for most practical purposes, especially if the base is fairly close to the ground. However, when the base of the machine is pitched with respect to the ground, a strong cross-flow establishes itself, with an unsymmetrical pattern, and strong destabilizing pitching and rolling moments may appear. The seriousness of this phenomenon depends obviously upon the amount of air which is being entrained by the peripheral annular jet.

In a very elegant first approximation, Chaplin assumed (Reference 1) that the amount of flow entrained by a curved jet was the same as that of a straight jet, for which both theoretical and experimental data are available (References 2 and 3). The purpose of the present investigation is to approach the problem analytically, by extending the classical straight jet flow theory to the curved jet.

A formulation of the problem was made earlier by Boehler in Reference 4 for a laminar jet and by Chang in Reference 5 for a turbulent jet. Here, the turbulent case is considered, since it is well known that all jet flows obtained experimentally are mostly turbulent. Formally, the equations of the present problem are the same as those of Reference 5. However, an important difference with Reference 5 is that the systems of axes are defined differently. In all cases, the basic assumption is made of the similarity of the velocity profiles. This assumption is restrictive, and may turn out in the long run not to be justified. However, it is a reasonable one for a first approach to the problem, since it makes possible a closed-form solution of the problem and allows a simple direct comparison with the straight jet case. Restrictive as it may be, the assumption of similarity was also used in the analytical solution of the boundary layer along a curved wall (Reference 6).

The solution is obtained by reducing the Navier-Stokes partial differential equations of the curved jet flow to a third-order total differential equation of the Hartree-Skan type. This equation is integrated numerically on the IBM 7090 computer using a Runge-Kutta subroutine. The boundary conditions are carefully established and discussed. The complete program for the numerical solution is included in the report. Typical numerical results, in the form of velocity and pressure distribution across the jet are presented and discussed.

The solution of the basic curved viscous jet is important not only for an understanding of the flow mechanics of ground effect machines, but for many other problems as well, such as those involving cavity flows (base pressure of missile boosters for example) or the understanding of the Coanda effect.

The two-dimensional solution can be extended to three dimensions, if desired, by means of a Mangler transformation (Reference 4).

I EQUATION OF THE PROBLEM

Historically, it was found (Reference 2, page 143 for example) that the equations governing the flow of a two-dimensional jet emerging from a narrow slit into a fluid at rest are formally the same as the boundary layer equations along a straight wall (with no pressure gradient); actually the two-dimensional jet case is one of the few cases where the Navier-Stokes equations can be integrated exactly.

Similarly, the equations of motion of a two-dimensional curved viscous jet are formally the same as those of the boundary layer along a curved wall. The general Navier-Stokes equations for the flow along a curved wall and their simplification by use of the standard boundary-layer approximations are discussed in detail in Reference 6. These equations are directly usable for the two-dimensional curved jet, as pointed out in Reference 4. One difference, however, lies in the definition of the system of axes and coordinates. This point bears a detailed discussion which will be made here.

For the boundary-layer flow along a curved wall, the choice of the coordinate system is obvious (figure 1 a). It is a curvilinear system of coordinates in which the x axis is in the direction of the wall and the y axis is normal to it. The stagnation point is taken as origin of coordinates. The curvilinear net therefore consists of curves parallel to the wall and of straight lines perpendicular to them. The corresponding velocity components are called u and v and the curvature at a point x is K(x), which is a continuous function of x and is positive for convex and negative for concave curvature. For the laminar case and for moderate wall curvatures, it is shown in Reference 6 that the boundary layer equations reduce to:

$$
I \left\{ \begin{aligned}
& u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \\
& \frac{\partial p}{\partial y} = \rho K u^2 \\
& \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\end{aligned} \right.
$$

Similarly, for the turbulent case, the equations of motion reduce to:

$$
\mathrm{II}\left\{
\begin{aligned}
& u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \epsilon\frac{\partial^2 u}{\partial y^2} \\
& \frac{\partial p}{\partial y} = \rho K u^2 \\
& \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\end{aligned}
\right.
$$

ϵ is the kinematic viscosity which is assumed to be determined from Prandtl's mixing-length theory (Reference 2, Chapter XIX). In general:

$$\epsilon = \chi_1 b (u_{max} - u_{min})$$

here: (III) $\epsilon = \chi_1 b U$

where U denotes the maximum velocity, b the width of the mixing zone and χ_1 is an empirical constant. ϵ is assumed to be independent of y.

Denoting the center-line velocity and the width of the jet at a fixed characteristic distance s from the orifice by U_s and b_s respectively, we may write:

$$U = U_s \left(\frac{x}{s}\right)^{-\frac{1}{2}} \quad ; \qquad b = b_s \frac{x}{s}$$

Consequently:

$$\epsilon = \epsilon_s \left(\frac{x}{s}\right)^{\frac{1}{2}} \quad \text{with} \quad \epsilon_s = \chi_1 b_s U_s$$

Further, the assumption of similarity results in the turbulent case in:

(IV) $\eta = \sigma \frac{y}{x}$

where σ is a free constant.

We may now come back to the problem at hand, i. e. the curved jet (fig. 1 b). The main difficulty in trying to apply the above boundary layer equations lies in the definition of a system of coordinates. One is first led to separate the jet into two regions: the "inside" jet and the "outside" jet. The line which separates the two regions is then defined as the "axis" of the jet. At the outset, the shape of the axis is not known. It will be shown later on that, out of many possible such axes, only one must be chosen, for no better reason than because that particular choice leads to a possible reduction of the Navier-Stokes partial differential equations to a single total differential equation. Thus a two-dimensional curved jet flow is the sum of two flows respectively analogous to the boundary-layer flow along a concave wall of curvature - R and that along a convex wall of curvature + R. For each region, then, using the proper sign of R, the above equations I and II hold.

One would be tempted to say that the maximum velocity of the flow U defined before in connection with the definition of the kinematic viscosity (equation III) is the center-line velocity, i. e. the velocity along the "axis" of the jet, as is the case for the straight jet. We shall see later that this is <u>not</u> the case. Indeed, one could arbitrarily define the axis as the line of maximum velocity; however, this condition would be insufficient to determine the equation of the axis and would lead to analytical difficulties in the solution of the problem.

By analogy with equation (IV) of the boundary layer case, the assumption of similarity of the velocity profiles is used in the solution of the curved jet case. The hypotheses of Prandtl's mixing-length theory are also assumed to hold, possibly with different values of the arbitrary constants, so that the maximum velocity U still varies like the inverse of the square root of x.

In the algebraic calculations involved in passing from the Navier-Stokes partial differential equations to the total differential equation, it is sufficient to consider only one-half of the problem, i. e. either the inside or the outside jet, since the other half can be obtained by changing the sign of the radius of curvature in the equations.

The problem of the curved viscous jet can therefore be stated as follows:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \epsilon \frac{\partial^2 u}{\partial y^2} \qquad (1)$$

$$\frac{\partial p}{\partial y} = \rho K u^2 \qquad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (3)$$

$$\epsilon = \chi_1 b_s U_s \left(\frac{x}{s}\right)^{\frac{1}{2}} \qquad (4)$$

$$\eta = \sigma \frac{y}{x} \qquad (5)$$

To solve these equations, one can, as for the straight jet, define a stream function, Ψ which is assumed to be of the form:

$$\Psi = \sigma^{-1} U_s s^{\frac{1}{2}} x^{\frac{1}{2}} F(\eta) \qquad (6)$$

$F(\eta)$ is the unknown function, the determination of which will give the solution of the problem.

Expressing the velocity components in terms of F and F' and substituting into the momentum equations, one can replace the partial differential equations by one total differential equation involving the function F. It must be noted that, in the following derivations, some liberties have been taken with respect to the notations for partial differentiation. These equations have been checked very carefully, however, and are believed to be correct. The calculations are as follows:

$$u = \frac{\partial \Psi}{\partial y} = \frac{\partial \Psi}{\partial \eta} \frac{\partial \eta}{\partial y} = U_s \left(\frac{s}{x}\right)^{\frac{1}{2}} F'(\eta) \qquad (7)$$

$$v = -\frac{\partial \Psi}{\partial x} = \frac{U_s}{\sigma}\left(\frac{s}{x}\right)^{\frac{1}{2}} \left[\eta F'(\eta) - \frac{F(\eta)}{2} \right] \qquad (8)$$

from equations (7) and (8), the left-hand member of equation (1) can easily be calculated.

Next one must calculate $\frac{\partial p}{\partial x}$, using equation (2) and substitute the value thus found into the right-hand member of equation (1). In keeping with the general assumption of similar velocity profiles, one will assume that y only appears in the expression of p through the term η . Hence:

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \eta} \frac{\partial \eta}{\partial x} \qquad (9)$$

Further:

$$\frac{\partial p}{\partial \eta} = \frac{\partial p}{\partial y} \frac{\partial y}{\partial \eta} \qquad (10)$$

From equation (2):

$$\frac{\partial p}{\partial y} = K \rho u^2$$

From equation (7):

$$u = U_s \left(\frac{s}{x}\right)^{\frac{1}{2}} F'(\eta) \qquad (11)$$

We stated earlier that y could not appear formally in the expression of p; similarly, it cannot appear in the expression of $\frac{\partial p}{\partial \eta}$. Therefore, equation (11) is only meaningful if the product Ky is a function of η alone. Rather than attempting to solve the problem for any function of η , one can choose a particular one, the simplest one, for example a linear function of η . One is therefore led to put:

$$Ky = \frac{A}{2} \eta \qquad (12)$$

where A is a constant.

or: $$2Ky = A\eta \qquad (12\,b)$$

By substitution, equation (11) simplifies to:

$$\frac{\partial p}{\partial \eta} = \rho U_s^2 \frac{s}{x} \frac{A}{2} F'^2(\eta) \qquad (13)$$

Note that, now, equation (12) defines the center line of the jet.

By substitution, one finds the following differential equation for F:

$$(A\eta + 1) F'^2 + F F'' + \frac{1}{2} F''' = 0 \quad , \qquad (14)$$

after letting:

$$\sigma = \frac{1}{2} \sqrt{\frac{U_s s}{\epsilon_s}} \qquad (15)$$

which is legitimate since X_1 in the expression of ϵ_s is a free constant.

Naturally, from equation (12), if $A = 0$, the curvature K of the jet is also zero. $A = 0$ corresponds to the straight jet case. Equation (14) reduces then to:

$$F'^2 + F F'' + \frac{1}{2} F''' = 0 \qquad (16)$$

Equation (16) is the differential equation of the straight jet solution (Reference 2, page 498), as it should be.

The solution of the curved jet problem amounts to finding a solution of equation (14), which also satisfies the proper boundary conditions. This solution must be referred to a system of axes which still remains to be determined. What one knows at this point is that the axes are defined by equation (12 b): $2Ky = A\eta$. Before proceeding with the solution, we shall determine the center line of the jet and these axes analytically. Obviously, the center line will consist of a family of curves depending upon one arbitrary parameter A.

II SHAPE OF THE JET

The shape of the jet is defined by the equation

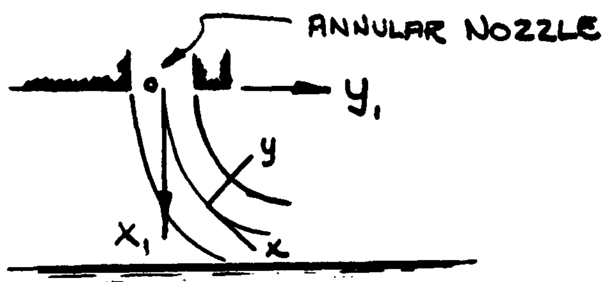
$$Ky = \frac{A}{2}\eta$$

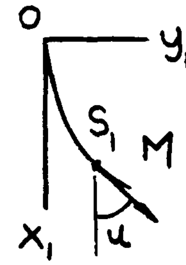
where: x and y are curvilinear coordinates along the curve, as shown on the figure

K is the curvature of the center line of the jet

$\eta = \sigma\, y/x$, where σ is a constant

A is a parameter





When one looks for the equation of the shape of the center line of the jet, which we have called x-axis, one must find this shape with respect to a fixed system of rectangular coordinates x_1 and y_1. The unknown equation of the curve is $y_1 = y_1(x)$ and is defined by the following intrinsic equation (since x becomes s_1, length of an arc on the curve):

$$Ky = A/2\ \sigma\,(y/s_1)$$

or $$K = A\ \sigma/2\ (1/s_1)$$

Defining a new constant $C = A\,\sigma/2$, one has:

$$K = C/s_1$$

Therefore, one wishes to find the curves for which the curvature at a given point is inversely proportional to the length of the arc between the origin and that point, the length of the arc being defined by the differential relationship:

$$ds_1^2 = dx_1^2 + dy_1^2$$

For simplification, we shall drop the suffix "1" in all that follows. Let us remember, though, that the curve to be found will be referred to the fixed system of coordinates $x_1 0 y_1$.

The radius of curvature $R = 1/K$, is given by

$$R = \frac{(1 + y'^2)^{3/2}}{y''} \qquad (17)$$

The intrinsic equation is:

$$K = C/s \qquad (18)$$

Therefore,

$$R = (1/C)s \qquad (19)$$

Differentiating with respect to x, there comes:

$$\frac{dR}{dx} = \frac{1}{C}\frac{ds}{dx} \qquad (20)$$

but

$$\frac{ds}{dx} = \sqrt{1 + y'^2} \qquad (21)$$

and

$$\frac{dR}{dx} = \frac{3/2\left[1 + y'^2\right]^{1/2} \times 2y'y''^2 - y'''\left[1 + y'^2\right]^{3/2}}{y''^2} \qquad (22)$$

or:

$$\frac{dR}{dx} = \frac{3\left[1 + y'^2\right]^{1/2} \times y'y''^2 - y'''\left[1 + y'^2\right]^{3/2}}{y''^2} \qquad (23)$$

Substituting (21) and (23) into equation (4), there comes:

$$3(1 + y'^2)^{1/2} y'y''^2 - y''' (1 + y'^2)^{3/2} = 1/C (1 + y'^2)^{1/2} y''^2$$

Simplifying, one finds:

$$3y'y''^2 - y''' (1 + y'^2) = 1/C\, y''^2 \qquad (24)$$

The problem is reduced to solve equation (24), a third order differential equation, the solution of which gives $y = y(x)$

Note that the equation is homogeneous and that neither x, nor y appear in it. To solve it, let us first consider $y' = dy/dx$ as the variable and $y'' = d^2y/dx^2$ as the unknown function. Therefore, let

$y' = t$, independent variable

$y'' = p$, unknown function

One has:

$$y''' = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{dp}{dx} = \frac{dp}{dy'}\,\frac{dy'}{dx} = p'p$$

The above equation is therefore of the form:

$$F(t, p, p') = 0$$

Substituting, one finds:

$$3tp^2 - p'p(1 + t^2) - (1/C)p^2 = 0$$

Dividing by p, there comes:

$$3tp - p' - p't^2 - p/C = 0$$

or $$p' (1 + t^2) + p (1/C - 3t) = 0$$

or $$\frac{p'}{p} = \frac{3t - 1/C}{1 + t^2}$$

$$\text{Log } p = \int \frac{3t - 1/C}{1 + t^2} dt = \frac{3}{2} \int \frac{2tdt}{1 + t^2} - \frac{1}{C}\int \frac{dt}{1 + t^2}$$

$$\text{Log}\ p = 3/2\ \text{Log}\ (1 + t^2) - 1/C \tan^{-1} t + C_1 \tag{25}$$

going back to the original variables, one has:

$$\text{Log}\ y'' = 3/2\ \text{Log}\ (1 + y'^2) - 1/C \tan^{-1} y' \tag{26}$$

Equation (26) is a second-order differential equation which is again homogeneous, and in which y does not appear.

Consider now y as the variable and $y' = dy/dx$ as the unknown function

$$\frac{dy}{dx} = y' = p$$

hence: $$\frac{d^2y}{dx^2} = y'' = \frac{dy'}{dx} = \frac{dy'}{dy}\,\frac{dy}{dx} = p'p$$

Substituting into equation (26), one finds now:

$$\text{Log}\ (p\,p') = 3/2\ \text{Log}\ (1 + p^2) - 1/C \tan^{-1} p \tag{27}$$

or: $$\text{Log}\ p + \text{Log}\ p' = 2/3\ \text{Log}\ (1 + p^2) - 1/C \tan^{-1} p$$

$$\text{Log}\ \frac{dp}{dy} = \frac{3}{2}\ \text{Log}\ (1 + p^2) - \text{Log}\ p - \frac{1}{C} \tan^{-1} p$$

$$\frac{dp}{dy} = \frac{(1 + p^2)^{3/2}}{p}\ e^{-(1/C)\tan^{-1} p}$$

Therefore, inverting:

$$dy = \frac{p}{(1 + p^2)^{3/2}}\ e^{(1/C)\tan^{-1} p}\ dp$$

$$y = \int \frac{p}{(1 + p^2)^{3/2}}\ e^{1/C \tan^{-1} p}\ dp$$

to solve the integral, let us make the change of variable $\tan^{-1} p = u$
or: $p = \tan u$

$$dp = (1 + \tan^2 u)\, du$$

hence: $$y = \int \frac{\tan u}{(1 + \tan^2 u)^{3/2}} e^{u/C} (1 + \tan^2 u)\, du$$

Simplifying, there comes:

$$y = \int \frac{\tan u}{(1 + \tan^2 u)^{1/2}} e^{u/C}\, du$$

but $$\frac{1}{1 + \tan^2 u} = \cos^2 u$$

hence: $$y = \int \frac{\sin u}{\cos u} \cos u \; e^{u/C}\, du$$

$$y = \int \sin u \; e^{u/C}\, du \tag{28}$$

Equation (28) can be integrated (Burington, Equation 312):
(where $a = 1/C$, $b = 1$)

$$y = \frac{e^{u/C}}{1 + 1/C^2} (1/C \sin u - \cos u) \tag{29}$$

together with: $p = \tan u$

Going back to the original variables, one has:

$$\left\{ \begin{aligned} & y = \frac{e^{u/C}}{1 + 1/C^2} (1/C \sin u - \cos u) \\ & \frac{dy}{dx} = \tan u \end{aligned} \right. \tag{30}$$

Let us now differentiate the first equation (30) and substitute into the second one:

$$dy = e^{u/C} \sin u \, du$$

therefore:

$$dx = \frac{dy}{\tan u} = e^{u/C} \frac{\sin u}{\frac{\sin u}{\cos u}} du = e^{u/C} \cos u \, du$$

$$x = \int e^{u/C} \cos u \, du \tag{31}$$

Equation (31) can be integrated (Burington, Equation 314):

finding: $x = \frac{e^{u/C}}{1 + 1/C^2} \left((1/C)\cos u + \sin u \right)$

Therefore, the parametric representation of the desired curve is as follows:

$$\left.\begin{aligned} x &= \frac{e^{u/C}}{1 + 1/C^2} \left((1/C)\cos u + \sin u \right) - \frac{C}{1 + C^2} \\ y &= \frac{e^{u/C}}{1 + 1/C^2} \left((1/C)\sin u - \cos u \right) + \frac{C^2}{1 + C^2} \\ \frac{dy}{dx} &= \tan u \end{aligned}\right\} \tag{32}$$

This equation is plotted in Figure 2 for $C = 0.5$, $C = 1.0$, $C = 2$ and $C = 4$.

III PROCEDURE FOR SOLUTION

The equation to be solved is:

$$(A\,\eta + 1)\,F'^2 + F\,F'' + \frac{1}{2}\,F''' = 0 \qquad (14)$$

An approximate solution of this equation was given in Reference 5. Here, a numerical but exact solution is discussed, which makes use of the IBM 7090 digital computer.

The equation is solved using the Runge-Kutta method. Putting:

$$F' = G \qquad (33)$$

$$F'' = G' = H \qquad (34)$$

equation (14) becomes:

$$(A\,\eta + 1)\,G^2 + F\,H + \frac{1}{2}\,H' = 0$$

or:

$$H' = -\,2\left[(A\,\eta + 1)\,G^2 + F\,H\right] \qquad (35)$$

The system of equations (33) to(35) can then be solved numerically using the digital computer, after the initial boundary conditions are specified. A thorough discussion of the boundary conditions will now be made.

The thee boundary conditions suggested and used in Reference 5 are as follows:

$$\text{at } \eta = 0 \qquad F(0) = 0 \qquad (36)$$

$$\text{at } \eta = 0 \qquad F'(0) = G(0) = 1 \qquad (37)$$

$$\text{at } \eta = \infty \qquad F'(\infty) = G(\infty) = 0 \qquad (38)$$

Preliminary runs on the computer indicated that the above boundary conditions are neither correct, nor sufficient to solve the problem.

For these early runs, the following procedure was adopted: the numerical calculations start from $\eta = 0$. One needs therefore to know the initial conditions on the function F and its derivatives at $\eta = 0$, i. e.:

$$F(0),\ G(0),\ H(0)$$

Equations (16) and (17) give:

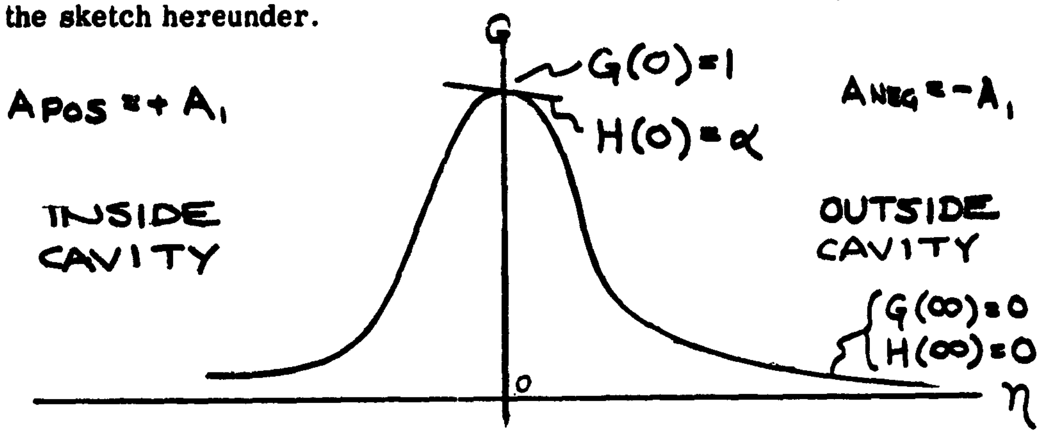
$$F(0) = 0$$
$$G(0) = 1$$

There remains to assume: $H(0) = \alpha$

From equation (7), if one plots u, x-component of the velocity, against η, $H(0) = \alpha$ represents the slope at $\eta = 0$ of that velocity distribution. For the straight jet, $H(0) = 0$ (the velocity profile is symmetrical with respect to $y = 0$). For the curved jet, H(0) can have any value between 0 and ∞. For small curvatures, however, it stands to reason that H will be small. For example, for $A = -0.25$, one found $H \approx -0.13$.

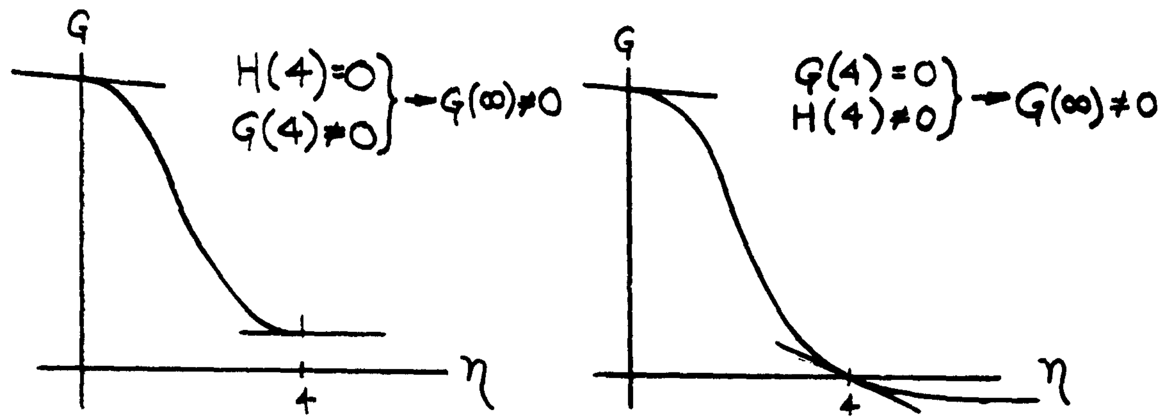
One can then start from an arbitrary value $H_1(0) = \alpha_1$ and integrate the equations (33) to (35) numerically. It was found that increments of η equal to 0.02 were satisfactory. In principle, one integrates until $\eta = \infty$. In practice, for computational purposes, "infinity" can be defined as the value of η for which $G(\eta) = 0 \pm 0.000\,005$. It was further found that, instead of using a variable $\eta\ (\infty)$ for which the above criterion would be satisfied, it was just as accurate and simpler to carry out all calculations up to $\eta = 4$ and take infinity at $\eta = 4$. This viewpoint was carried out in all subsequent calculations. An additional difficulty which occurred because of the use of $\eta = 4$ in lieu of "infinity" is discussed later on. The trial values $\alpha_1 \alpha_2 \alpha_3$ of α can be adjusted until the boundary condition (38) at infinity is satisfied with the desired accuracy.

The printout of early programs consisted of four columns of numbers showing respectively η, F, G and H for all values of η between 0 and 4 with ($\Delta\eta = 0.02$). The velocity distribution across the jet in each case could be obtained by plotting G against η, as shown in the sketch hereunder.



It immediately became apparent that the three boundary conditions (36) to (38) were not sufficient, but that one also needed to have the boundary condition $H(\infty) = 0$ (i. e. the slope of the velocity profile at infinity had to be zero), in order to have physically-possible velocity profiles. Thus, it would appear at first sight that one had now four boundary conditions: $F(0) = 0$, $F'(0) = 1$, $G(\infty) = 0$, $H(\infty) = 0$ and that the problem would be overdetermined. Such is not really the case. The difficulty is due to the fact that one had to replace infinity by a finite value of η ($\eta = 4$), in order to save lengthy step-by-step integrations. It turns out that the boundary condition $G(\infty) = 0$ is, for numerical purposes, equivalent to the two conditions: $G(4) = 0$, $H(4) = 0$. If one were to be at infinity, $G(\infty) = 0$ would automatically entrain $H(\infty) = 0$, which therefore would not have to be prescribed as part of the problem.

At $\eta = 4$, if one were to specify only one of the two conditions: $G(4) = 0$, $H(4) = 0$, either of the two situations pictured hereunder could take place, both being physically unacceptable:



In all subsequent calculations, it was therefore accepted that the boundary conditions of the problem at infinity could be replaced by $G(4) = 0$, $H(4) = 0$. In fact, one sometimes had $G(4) = 0$, but one never could have $H(4) = 0$. One usually had to be satisfied with $H(4) \leq 0.003$.

The next step in the work was to set up an iteration procedure for $H(0) = \alpha_i$. This could be done simply by assuming two initial values α_1 and α_2 and setting up an interpolation formula from which the desired value of α could be determined. This formula is not discussed here, but was included in the first program, which is reproduced in part IV of this report. Convergence was very rapid (usually 2 or 3 iterations).

During the early runs, an alternate method of approach was tried. The iteration was carried out backwards from $\eta = 4$ to $\eta = 0$. The initial boundary conditions were then:

$F(4) = \beta$, assumed value
$G(4) = 0$
$H(4) = 0$

The value of β was changed until a value was found which satisfied the boundary condition $G(0) = 1$.

This method turned out to be unsatisfactory for many reasons. At best, it had no marked advantage over the direct integration method.

At worse, it was very inaccurate, because the result depended greatly upon the assumed values of G(4) and of H(4). If one assumed $G(4) = H(4) = 0$, the numerical integration would never get "started", because these conditions could only be satisfied at infinity. One had to assume, for example, $H(4) = 0.000\,005$. Then, it turned out that the whole solution depended greatly upon the assumed value of H(4). The backwards integration was therefore abandoned.

To summarize these early investigations, for any given value of a parameter A, the equation:

$$(A\,\eta + 1)\,F'^2 + F\,F'' + \frac{1}{2}\,F''' = 0$$

subject to the boundary conditions:

$$F(0) = 0$$
$$G(0) = 1$$
$$H(\infty) = 0$$

gives a unique solution: $F(\eta)$, $G(\eta)$, $H(\eta)$. This solution is obtained by determining by an iteration formula the quantity $\alpha = H(0)$, such as to satisfy simultaneously: $G(4) \approx H(4) \approx 0$.

After a number of runs were made, it became apparent that the above formulation of the problem is <u>not correct</u>. Naturally, in retrospect, this became obvious: it is interesting to see how a numerical investigation can lead to a better theoretical understanding of a problem! None of the statements made above are wrong; however, an important boundary condition has been omitted; that which expresses the matching on the axis of the "inside" and of the "outside" jet. For a given value of the constant A, say A_1, the inside of the jet is found as solution of the equation:

$$(A_1\,\eta + 1)\,F'^2 + F\,F'' + \frac{1}{2}\,F''' = 0\ ;$$

the outside of the jet is found as solution of the equation:

$$(-A_1 \eta + 1) F'^2 + F F'' + \frac{1}{2} F''' = 0$$

These two equations are obviously distinct. If one solves them together with the boundary conditions (36) to (38), one must now express in addition that the slope $H_i(0)$ of the velocity distribution for the inside jet is equal in absolute value to the slope $H_o(0)$ of the velocity distribution for the outside jet. Numerical results as well as intuition tell us that there is no reason why this condition should turn out to be satisfied automatically; the problem as stated above is therefore overdetermined.

A reexamination of the hypotheses indicated that the problem can be stated correctly for the complete curved jet by eliminating boundary condition (36): $F(0) = 0$. This condition was written initially by analogy with the straight jet case, in which case it was a consequence of symmetry, since, for the straight jet, $v = 0$ on the axis (there is no cross flow across the axis). Because of the formal resemblance between the equations for straight and for curved jet, here again $F(0) = 0$ is equivalent to have $v = 0$ at $\eta = 0$, i. e. on the axis (this can be seen immediately from equation (8)). This condition would be justified if the axis was a streamline. However, as explained before, the axis of the jet (line $\eta = 0$) is defined by the condition $2 K y = A \eta$, and this does not make it a streamline. The boundary condition $F(0) = 0$ can therefore be dropped without inconsistency and replaced by the condition that

$$H(0)\Big]_{\substack{\text{inside} \\ @\,\eta = 0}} = H(0)\Big]_{\substack{\text{outside} \\ @\,\eta = 0}} \qquad (39)$$

In the remaining part of the investigation, the boundary conditions (37) to (39) were therefore used in connection with the numerical solution of equation (14). The complete solution was obtained in two steps, which will now be described in detail.

To obtain a numerical solution of the problem, it is still necessary to start with numerical values of F(0), G(0) and H(0). As before, G(0) = 1 and H(0) can be obtained by iteration, assuming initially two arbitrary values α_1 and α_2, It would have been possible to set up an iteration procedure for F(0) similar to that developed for H(0). Instead, it was found simpler to find the proper value of F(0) graphically. A first program was written, simply to establish in all cases the proper initial values F(0), G(0), H(0). The graphical procedure is illustrated in figure 3, corresponding to the following initial conditions:

$$\begin{cases} F(0) = 0.001,\ 0.005,\ 0.01 \text{ successively} \\ G(0) = 1 \\ F'(\infty) = 0 \\ \qquad\qquad A = +\,0.25; \qquad A = -\,0.25 \end{cases}$$

In each of the 6 cases above, equation (14) was integrated numerically and H(0) found by iteration. The result is plotted in figure 3 in the form of F(0) against H(0). In this, as in all subsequent cases, it was found that F(0) varied linearly with H(0). The value of F(0) corresponding to the solution of the complete jet is found, according to boundary condition (39), as the intersection of the two lines A = + 0.25 and A = - 0.25 on figure 3.

The above calculations were repeated for a number of values of A between 0.025 and 0.50. Some of the results are shown in figure 4. Next, the solutions of the curved jet problem F(0) and H(0) found as described above were plotted against the parameter A The results are summarized in figure 5. One startling result came out unequivocally: H(0) varies linearly with A; i. e. the slope of the velocity distribution curve at $\eta = 0$ varies linearly with the parameter A which determines the shape of the jet.

The variations of F(0), G(0), H(0) as a function of A are interesting to follow on figure 5, starting from A = 0. A = 0 corresponds to the straight jet; in this case, as we know, F(0) = 0 , G(0) = 1, H(0) = 0. As A increases, F(0) becomes negative while decreasing, then starts increasing, becomes zero and next becomes positive. This means that the velocity v across the axis of the jet is positive for small curvatures and becomes negative for large curvatures (this comes out of equation (8)). As A increases, G(0) remains always equal to 1 and, as noted before, H(0) increases linearly with A.

Figure 5 therefore completely depicts all initial conditions of the problem which automatically satisfy the boundary conditions (37) to (39). The information of figure 5 can therefore be used in lieu of the boundary conditions, hence reducing the machine program to a straightforward numerical integration without iterations or cross-plots.

A second program was therefore prepared to perform the Runge-Kutta integration, for known initial values F(0), G(0), H(0). This program is listed in part IV of this report. Note that the information contained in the first program is also listed in the second one, but <u>is not used</u>. This was done so that, if somebody wished at a later date to obtain F(0) and H(0) with great precision, this could be done without having to revert to the first program. Therefore, the first program is listed in part IV for information only, since all that it contains is also contained in the second program.

Provisions were made in the second program to calculate the integral $\int_{\eta}^{\eta+\Delta\eta} G^2(\eta)\, d\eta$ and print the result. This allows to calculate the pressure distribution across the jet as follows:

From equation 13, an elemental pressure variation across the jet is given by:

$$dp = \rho U_s^2 \frac{s}{x} \frac{A}{2} G^2(\eta)\, d\eta$$

Remembering that: $U_s^2 \frac{s}{x} = U^2$,

one has: $dp = \frac{1}{2} \rho U^2 A G^2 d\eta$

The total pressure across the jet is therefore:

$$\Delta p = \frac{1}{2} \rho U^2 A \int_{-\infty}^{+\infty} G^2(\eta)\, d\eta \qquad (40)$$

IV PROGRAM

Two programs are shown here. The first program was used to obtain the data of figure 5. The second program is more general and includes the elements of the first program as an option.

All symbols used are defined; listings of the two programs are shown, as well as the data card input.

Symbolic location	Contents
A	Constant in differential equation (12)
ABSF	Closed subroutine used to compute the absolute value
AF	Initial value of F (i. e. F(0))
ALP (η)	Initial value of H (i. e. H(0))
DEL X	Increment of η (DEL X = 0.02)
EN	Maximum numerical value of η (EN = 4)
F (η)	Unknown function of the problem
F 1	η at specified interval of integration
G (η)	= F' (η)
G 1	Value of G from the previous integration step
H (η)	= G' (η)
II	index of values of A
IIJ	iteration counter
IIP	index variable for the value of F(0)
INT	closed subroutine which initiates the subroutine RWINT (SHARE distribution)
INTM	subsequent entries to RWINT Note: DAUX is used by RWINT to compute the derivatives G and H.
IOP	print option: if non zero print every case if zero, only print the converged value of H
JJ	variable index used to go from one value of H to the next one
K	index variable used in interpolation formula
M	the number values of A

Symbolic location	Contents
MM	Maximum number of iterations and maximum number of values read in for F(0)
OP	option
T(2)	η
T(3)	delta η
T(4)	F(0)
T(5)	G(0)
T(6)	H(0)

Additional symbols for the second program

G 11	G square
G 12	sum from η_1 to η_2 of G square dη
NN	option whether to compute G 12 or not
SIMPUN	Closed subroutine to evaluate the integral sum of G^2 dη (SHARE distribution).

FIRST PROGRAM

```
      DIMENSION A(100),T(100),F(1000),G(1000),H(1000),ALP1(100),ALP(1000
     1),F1(500)
     X,AF(100)
      COMMON A,T,F,G,H,II,ALP1,ALP
     X,F1
   16 FORMAT (78H0            ETA                 F                    G
     X          H                 )
      READ 10
   10 FORMAT (72H1
     X                          )
      PRINT 10
      READ 12,EN,M,MM,DELX,(ALP(JJ),JJ=1,2),IOP
   12 FORMAT (1F6.0,2I6,3F16.6,I6)
      READ 13, (A(I),I=1,M)
   13 FORMAT (4F16.6)
      READ 170,(AF(I),I=1,MM)
  170 FORMAT(4E16.4)
      PRINT 50,EN,M,MM,(ALP(JJ),JJ=1,2),DELX
   50 FORMAT (25H0 THE UPPER LIMIT ON ETA=F6.2/27H0 THE NUMBER OF VALUE
     XOF A=I6/30H0MAXIMUM NUMBER OF ITERATIONS=I6/22H0 INITIAL VALUES OF
     X H=2F16.6/12H0 STEP-SIZE=F16.6)
      PRINT 51
   51 FORMAT (1H0,30X,8HA-VALUES )
      PRINT 52,(A(I),I=1,M)
   52 FORMAT (30X,F16.6)
      II=1
      IF(IOP) 53,53,55
   55 SENSE LIGHT 2
   53 SENSE LIGHT 1
      JJ=1
      IIP=1
      K=1
      IIJ=1
      IF (EN) 14,14,15
   14 CALL ENDJOB
   15 T(2)=0.0
      T(3)=DELX
      T(4)=AF(IIP)
      T(5)=1.0
      T(6)=ALP(JJ)
      I=1
      CALL INT(T,3,1,0,0,0,0,0,0)
      F1(I)=T(2)
      F(I)=T(4)
      G(I)=T(5)
      H(I)=T(6)
      I=I+1
   77 CALL INTM
      F1(I)=T(2)
      F(I)=T(4)
      G(I)=T(5)
   19 H(I)=T(6)
      I=I+1
      IF(EN-T(2))18,77,77
   18 IF(ABSF(T(5))-.0005) 39,39,40
   39 SENSE LIGHT 0
      GO TO 22
   40 IF(SENSE LIGHT 1) 41,45
   41 JJ=JJ+1
```

```
      G1=T(5)
      IF (SENSE LIGHT 2) 21,15
   45 ALP( K+2)= (ALP(K+1)-(T(5)*(ALP(K)-ALP(K+1))/(G1-T(5))))
      JJ=K+2
      K=K+1
      G1=T(5)
      IIJ=IIJ+1
      IF(IIJ-MM) 56,56,80
   56 IF(SENSE LIGHT 2) 21,15
   80 PRINT 25,IIJ
   25 FORMAT (20X,I6,21H ITERATIONS COMPLETED)
      SENSE LIGHT 0
      IIJ=1
      GO TO 22
   21 SENSE LIGHT 2
   22 PRINT 10
      PRINT 16
      PRINT 17,A(II)
   17 FORMAT (1H0,40X,F16.6)
      I=1
   23 PRINT 31,F1(I),F(I),G(I),H(I)
      IF(EN-F1(I))30,30,32
   30 IF(SENSE LIGHT 2)33,35
   32 I=I+1
      GO TO 23
   33 CONTINUE
      SENSE LIGHT 2
      IF(IIJ-MM) 15,15,35
   35 IIP=IIP+1
      IIJ=1
      IF(IIP-MM) 190,180,180
 190  SENSE LIGHT 1
      SENSE LIGHT 2
      K=1
      JJ=1
      GO TO 15
 180  IIP=1
      IIJ=1
      II=II+1
      IF(A(II)) 191,192,192
 191  JJ=1
      ALP(JJ)=-ALP(JJ)
      ALP(JJ+1)=-ALP(JJ+1)
      GO TO 193
 192  JJ=1
      ALP(JJ)=ABSF(ALP(JJ))
      ALP(JJ+1)=ABSF(ALP(JJ+1))
 193  SENSE LIGHT 1
      SENSE LIGHT 2
      K=1
      JJ=1
      IF(II-M)15,15,14
   31 FORMAT (4F16.6)
      END
```

```
 SUBROUTINE DAUX
 DIMENSION A(100),T(100),F(1000),G(1000),H(1000),ALP1(100),ALP(1000
1),F1(500)
 COMMON A,T,F,G,H,II,ALP1,ALP
 T(7)=T(5)
 T(8)=T(6)
 T(9)=-2.((A(II)*T(2)+1.)*T(5)*T(5)+T(4)*T(6))
 RETURN
 END
```

SECOND PROGRAM

```
      DIMENSION A(100),T(100),F(1000),G(1000),H(1000),ALP1(100),ALP(1000
     1),F1(500)
     X,AF(100)
     X, G11(500),G12(500)
      COMMON A,T,F,G,H,II,ALP1,ALP
     X,F1
     X,G11,G12
16    FORMAT(120H0           ETA                  F                  G
     X           H           G(SQUARE)          SUM(G((SQUARE)XD ETA )      )
      READ 10
   10 FORMAT (72H1
     X                       )
      PRINT 10
      READ 121 ,M,MM,IOP,NN,DELX
121   FORMAT(4I6,F16.6)
      READ 12, EN,(ALP(JJ),JJ=1,IOP)
12    FORMAT (4F16.6)
      READ 13, (A(I),I=1,M)
   13 FORMAT (4F16.6)
      READ 170,(AF(I),I=1,MM)
170   FORMAT(4E16.4)
      PRINT 50,EN,M,MM,(ALP(JJ),JJ=1,2),DELX
   50 FORMAT (25H0 THE UPPER LIMIT ON ETA=F6.2/27H0 THE NUMBER OF VALUE
     XOF A=I6/30H0MAXIMUM NUMBER OF ITERATIONS=I6/22H0 INITIAL VALUES OF
     X H=2F16.6/12H0 STEP-SIZE=F16.6)
      PRINT 51
   51 FORMAT (1H0,30X,6HA-VALUES )
      PRINT 52,(A(I),I=1,M)
   52 FORMAT (30X,F16.6)
      II=1
      IF(IOP) 53,53,55
   55 SENSE LIGHT 2
   53 SENSE LIGHT 1
      JJ=1
      IIP=1
      K=1
      IIJ=1
      IF (EN) 14,14,15
   14 CALL ENDJOB
   15 T(2)=0.0
      T(3)=DELX
      T(4)=AF(IIP)
      T(5)=1.0
      T(6)=ALP(JJ)
      I=1
      CALL INT(T,3,1,0,0,0,0,0,0)
      F1(I)=T(2)
      F(I)=T(4)
      G(I)=T(5)
      H(I)=T(6)
      G11(I)=G(I)*G(I)
      I=I+1
   77 CALL INTM
      F1(I)=T(2)
      F(I)=T(4)
      G(I)=T(5)
   19 H(I)=T(6)
      G11(I)=G(I)*G(I)
      I=I+1
```

```
      IF(EN-T(2))1600,77,77
1600  IF(NN) 18,18,1801
1801  NN1=2
      DO 1802 I=1,NN
1802  G12(I)=SIMPUN(F1(I),G11(I),NN1)
      PRINT 10
      PRINT 16
      DO 1803 I=1,NN
1803  PRINT131,F1(I),F(I),G(I),H(I),G11(I),G12(I)
131   FORMAT(6F16.6)
      JJ=JJ+1
      IIP=IIP+1
      II=II+1
      IF(JJ-IOP)15,15,14
 18   IF(ABSF(T(5))-.0005) 39,39,40
39    SENSE LIGHT 0
      GO TO 22
  40  IF(SENSE LIGHT 1) 41,45
  41  JJ=JJ+1
      G1=T(5)
      IF (SENSE LIGHT 2) 21,15
  45  ALP( K+2)= (ALP(K+1)-(T(5)*(ALP(K)-ALP(K+1))/(G1-T(5))))
      JJ=K+2
      K=K+1
      G1=T(5)
      IIJ=IIJ+1
      IF(IIJ-MM) 56,56,80
  56  IF(SENSE LIGHT 2) 21,15
  80  PRINT 25,IIJ
  25  FORMAT (20X,I6,21H ITERATIONS COMPLETED)
      SENSE LIGHT 0
      IIJ=1
      GO TO 22
  21  SENSE LIGHT 2
  22  PRINT 10
      PRINT 16
      PRINT 17,A(II)
  17  FORMAT (1H0,40X,F16.6)
      I=1
  23  PRINT 31,F1(I),F(I),G(I),H(I)
      IF(EN-F1(I))30,30,32
  30  IF(SENSE LIGHT 2)33,35
  32  I=I+1
      GO TO 23
  33  CONTINUE
      SENSE LIGHT 2
      IF(IIJ-MM) 15,15,35
  35  IIP=IIP+1
      IIJ=1
      IF(IIP-MN) 190,180,180
190   SENSE LIGHT 1
      IF(IOP) 1900,1900,1901
1901  SENSE LIGHT 2
1900  K=1
      JJ=1
      GO TO 15
180   IIP=1
```

```
      IIJ=1
      II=II+1
      IF(A(II)) 191,192,192
191   JJ=1
      ALP(JJ)=-ALP(JJ)
      ALP(JJ+1)=-ALP(JJ+1)
      GO TO 193
192   JJ=1
      ALP(JJ)=ABSF(ALP(JJ))
      ALP(JJ+1)=ABSF(ALP(JJ+1))
193   SENSE LIGHT 1
      IF(IOP) 1933,1933,1930
1930  SENSE LIGHT 2
1933  K=1
      JJ=1
      IF(II-M)15,15,14
   31 FORMAT (4F16.6)
      END
```

Input

For the second program, the input data consists of the following cards:

Card No.	Columns	Contents
1	1	1
	2-72	Problem identification
2	1-6	M
	7-12	MM
	13-18	IOP
	19-24	NN
	25-40	DELX
3 to N	1-16	EN
	17-NN*	H(0)
N + 1 to M	1-NN	values of A
M + 1 to L	1-NN	values of AF

* The first value of H(0) is punched in Columns 17-32, and successive IOP values are punched; field width of 16 up to and including column NN; 4 words per card.

For the first program:

same as the second program, except

IOP = 2
NN = 0
leave cards N + 1 to L off.

V DISCUSSION OF RESULTS

As discussed earlier, the emphasis in the present work was placed more on the formulation and the solution of the problem than on the exploitation of the results. The program included in part IV of the report will allow anyone to generate immediately as many velocity and pressure distributions as he wishes.

Typical velocity and pressure distributions are shown in Figures 6 and 7 for several values of the curvature parameter A. The first figure shows a plot of $G(\eta)$ against η . This only represents the x-component of the velocity; in order to have the true velocity distribution, it would be necessary to add the y-component of the velocity. The plot is representative, however, since v is expected to be small compared to u. The velocity distribution plot shows that, as the curvature increases, the maximum velocity U increases and the location of the point of maximum velocity moves radially outwards (in the region of positive curvature, or, for an annular jet, towards the center of the cavity). This trend is in agreement with the experimental data on two-dimensional curved jets of Reference 7. Overall, however, the differences between straight and curved jet velocity distributions are not substantial (again, this could be modified due to the effect of the transverse velocity).

Figure 7 shows a plot of the calculated pressure distribution across the jet. It can be seen that the pressure variation is not exactly linear as the momentum theory would predict. Also, the pressure rise on both sides of the axis of the jet are not exactly equal.

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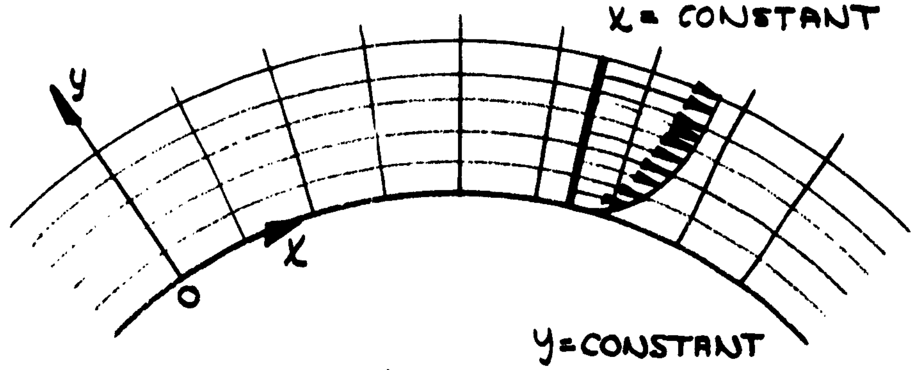


FIG. 1a COORDINATE SYSTEM FOR TWO-DIMENSIONAL BOUNDARY LAYER FLOW ALONG A CURVED WALL.

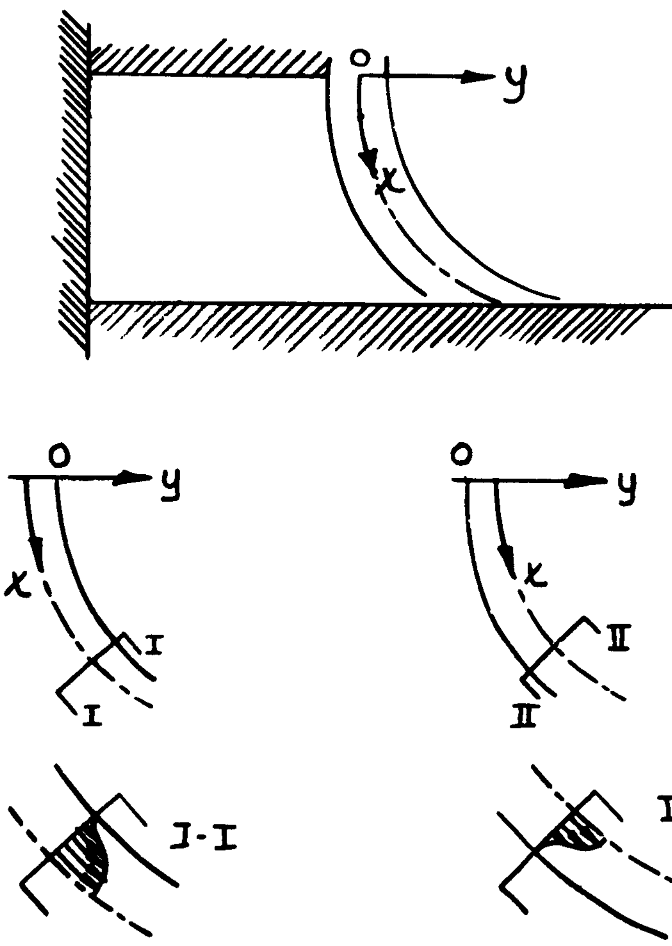


FIG. 1b COORDINATE SYSTEM FOR A CURVED JET

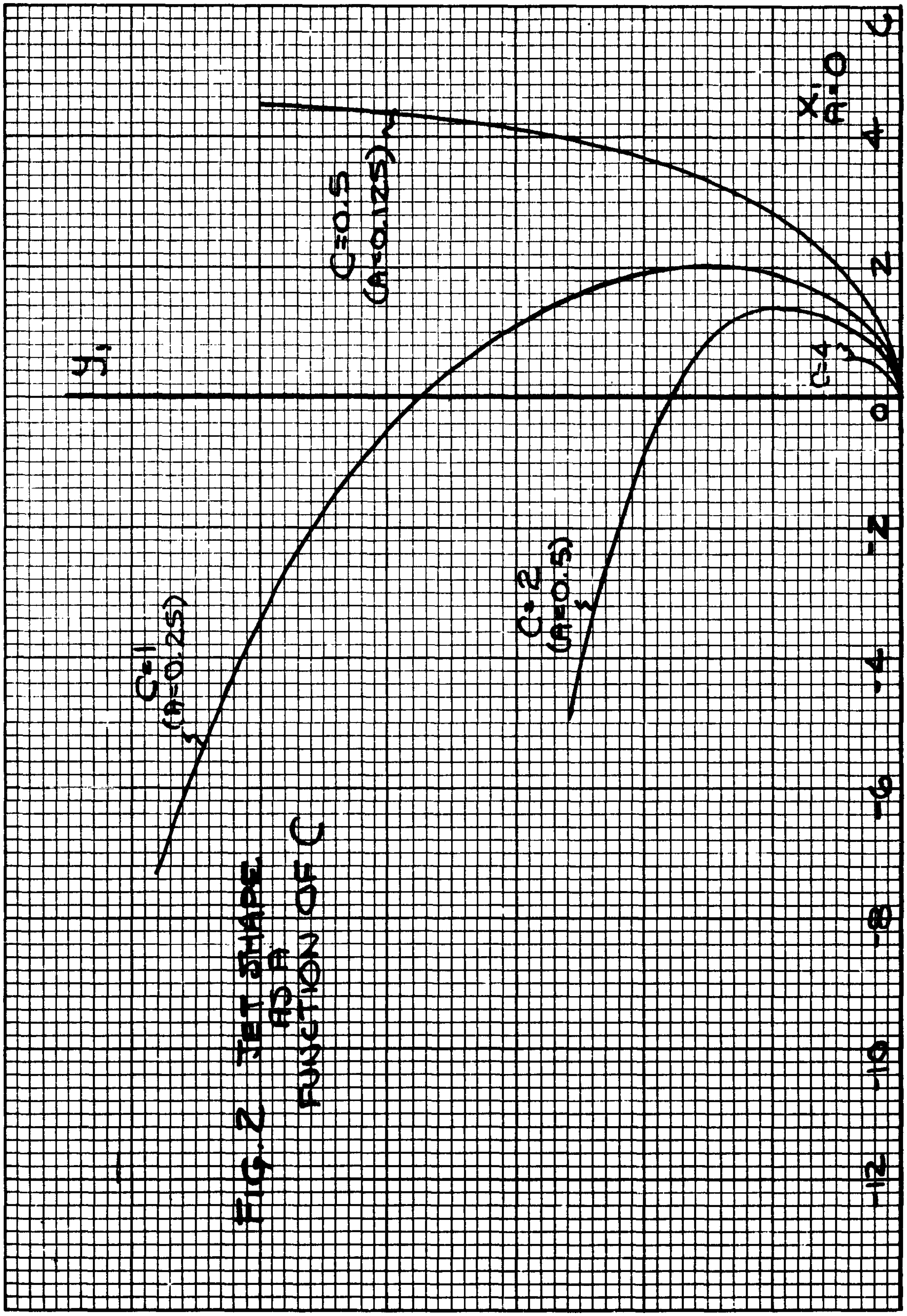


FIG. 2 JET SHAPE AS A FUNCTION OF C

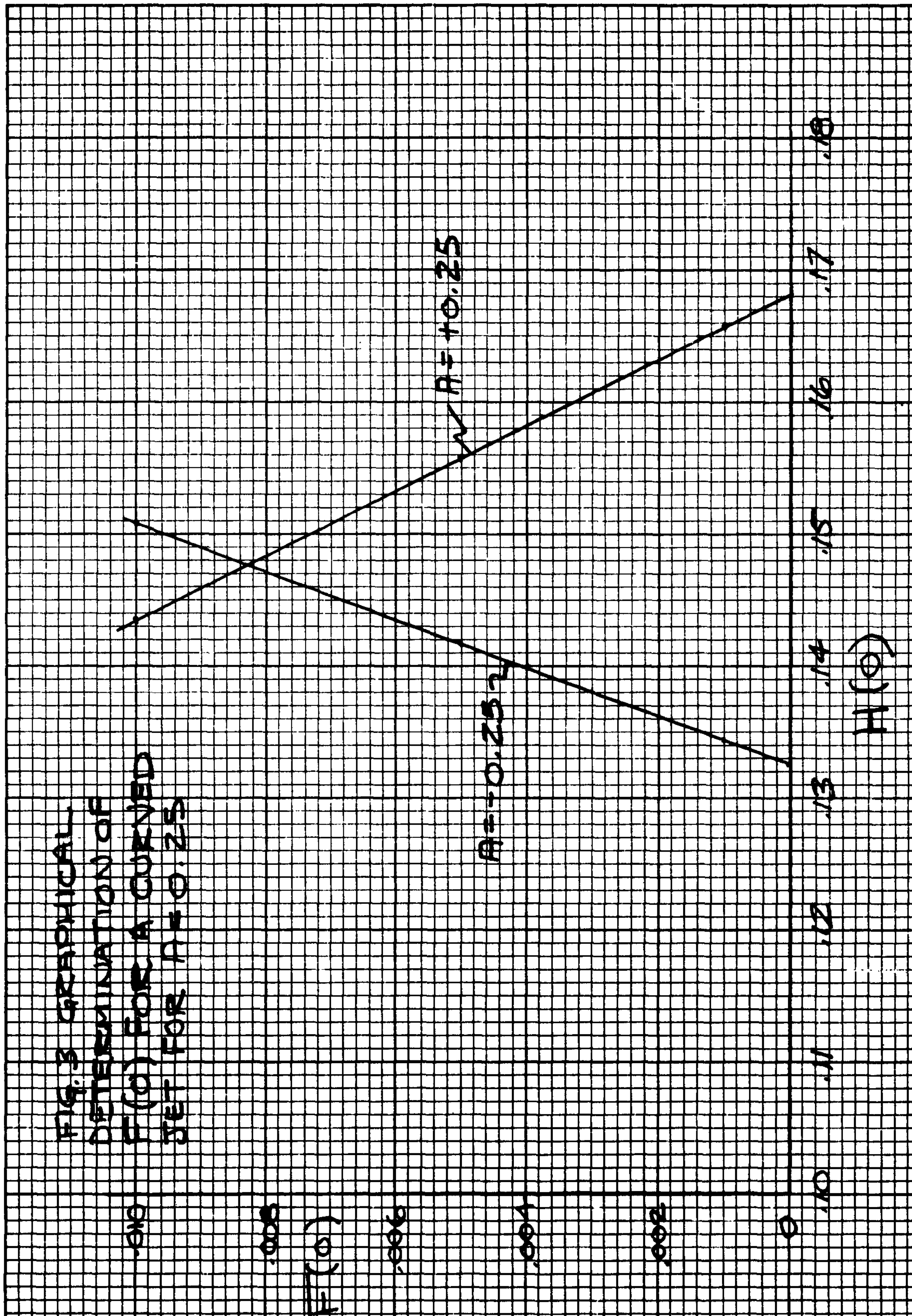
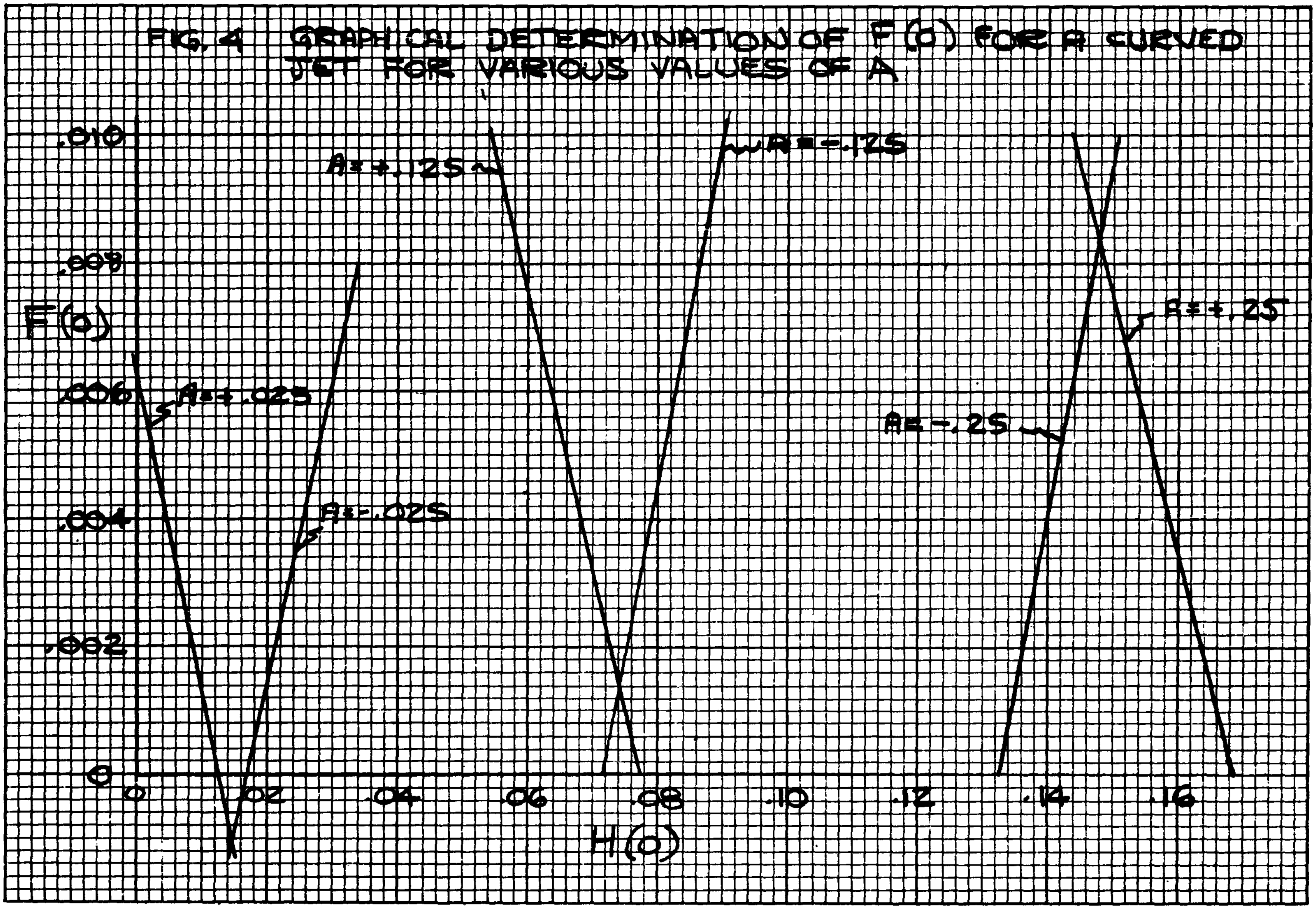


FIG. 3 GRAPHICAL DETERMINATION OF F(o) FOR A CURVED JET FOR A=0.25

FIG. 4 GRAPHICAL DETERMINATION OF F(o) FOR A CURVED JET FOR VARIOUS VALUES OF A



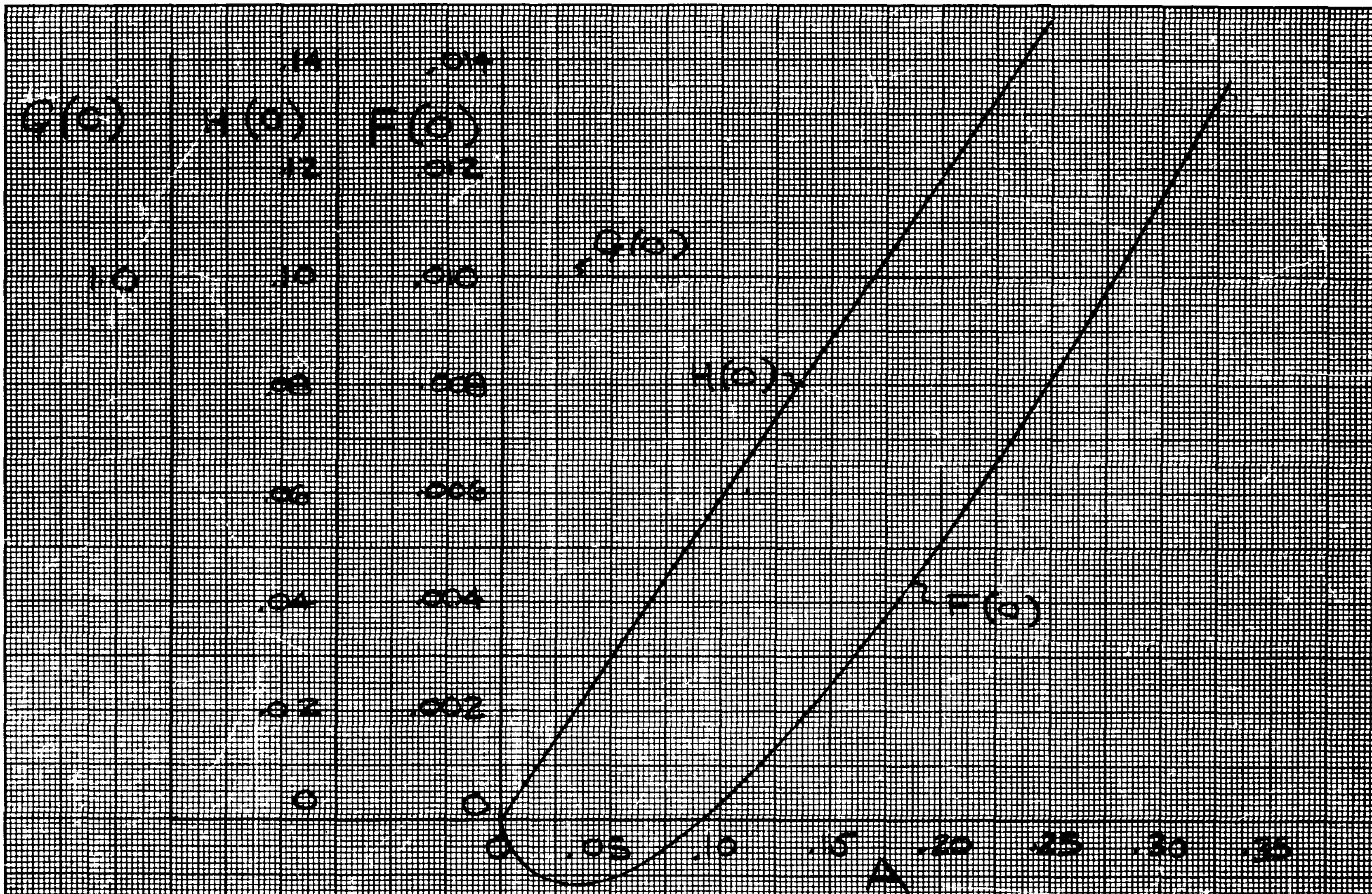


FIG. 5 VARIATION OF INITIAL CONDITIONS FOR A CURVED JET AS A FUNCTION OF SHAPE PARAMETER A

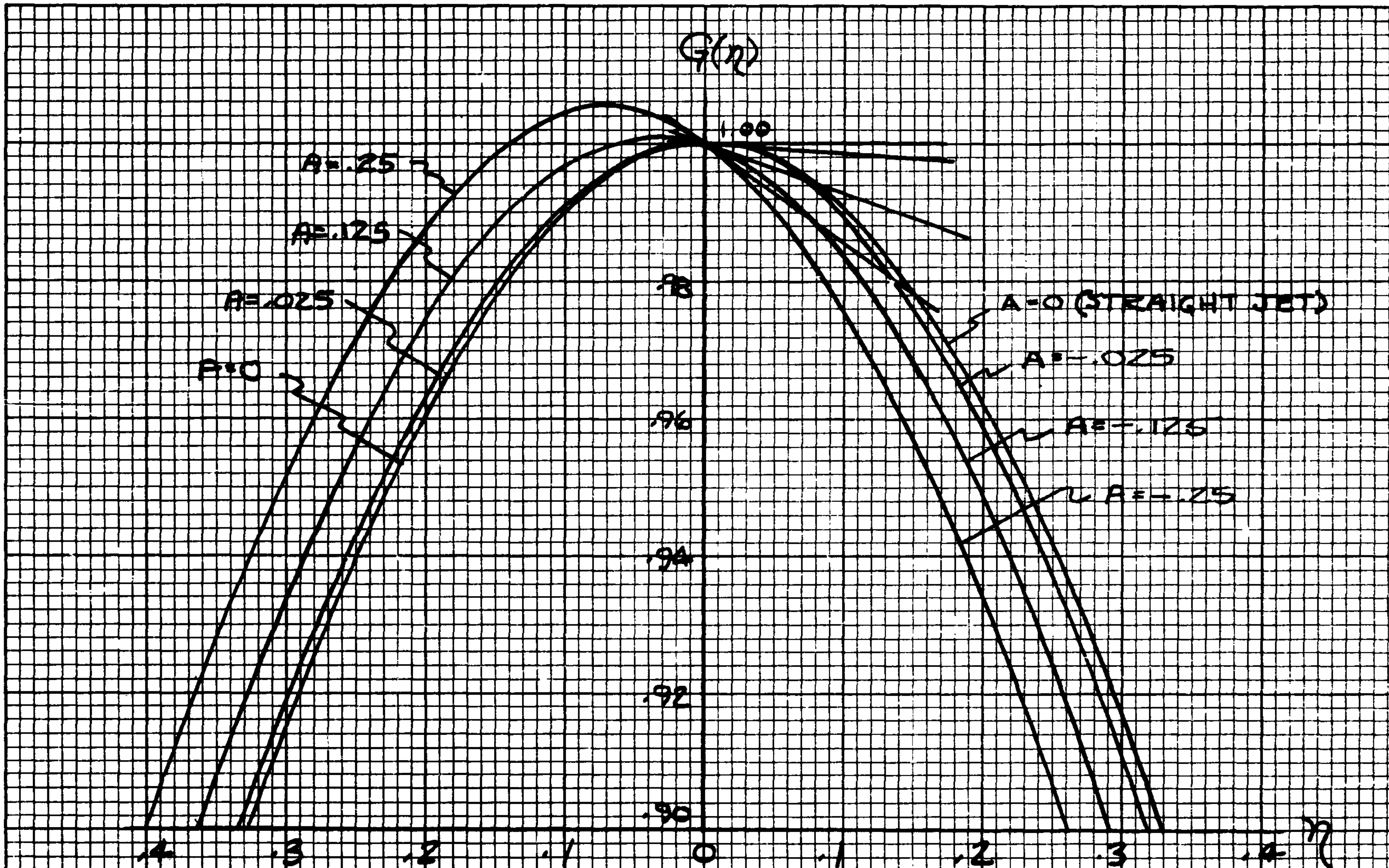


FIG. 6a EFFECT OF CURVATURE ON X- COMPONENT OF VELOCITY DISTRIBUTION OF CURVED JET.

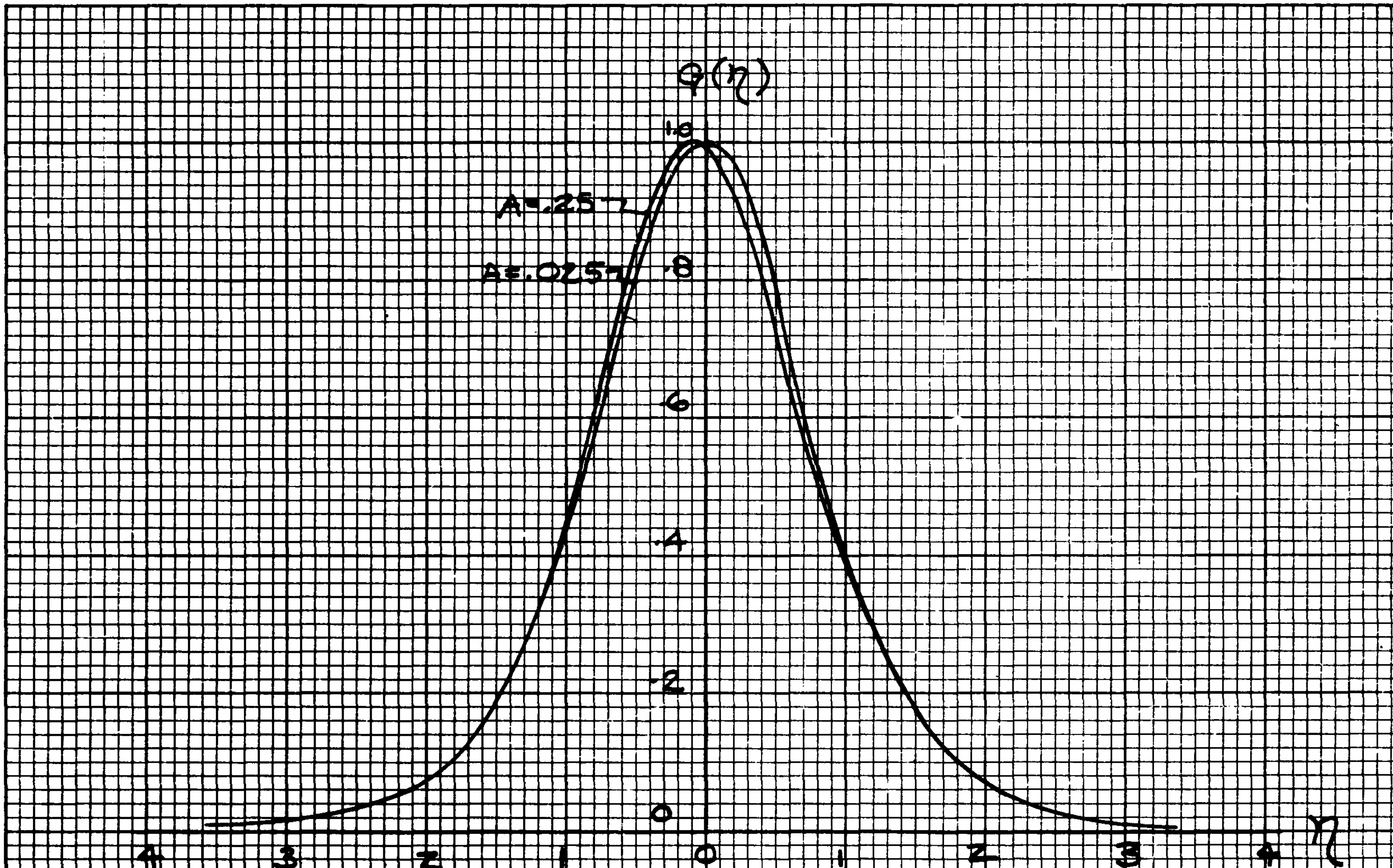


FIG. 6b EFFECT OF CURVATURE ON X- COMPONENT OF VELOCITY DISTRIBUTION OF CURVED JET

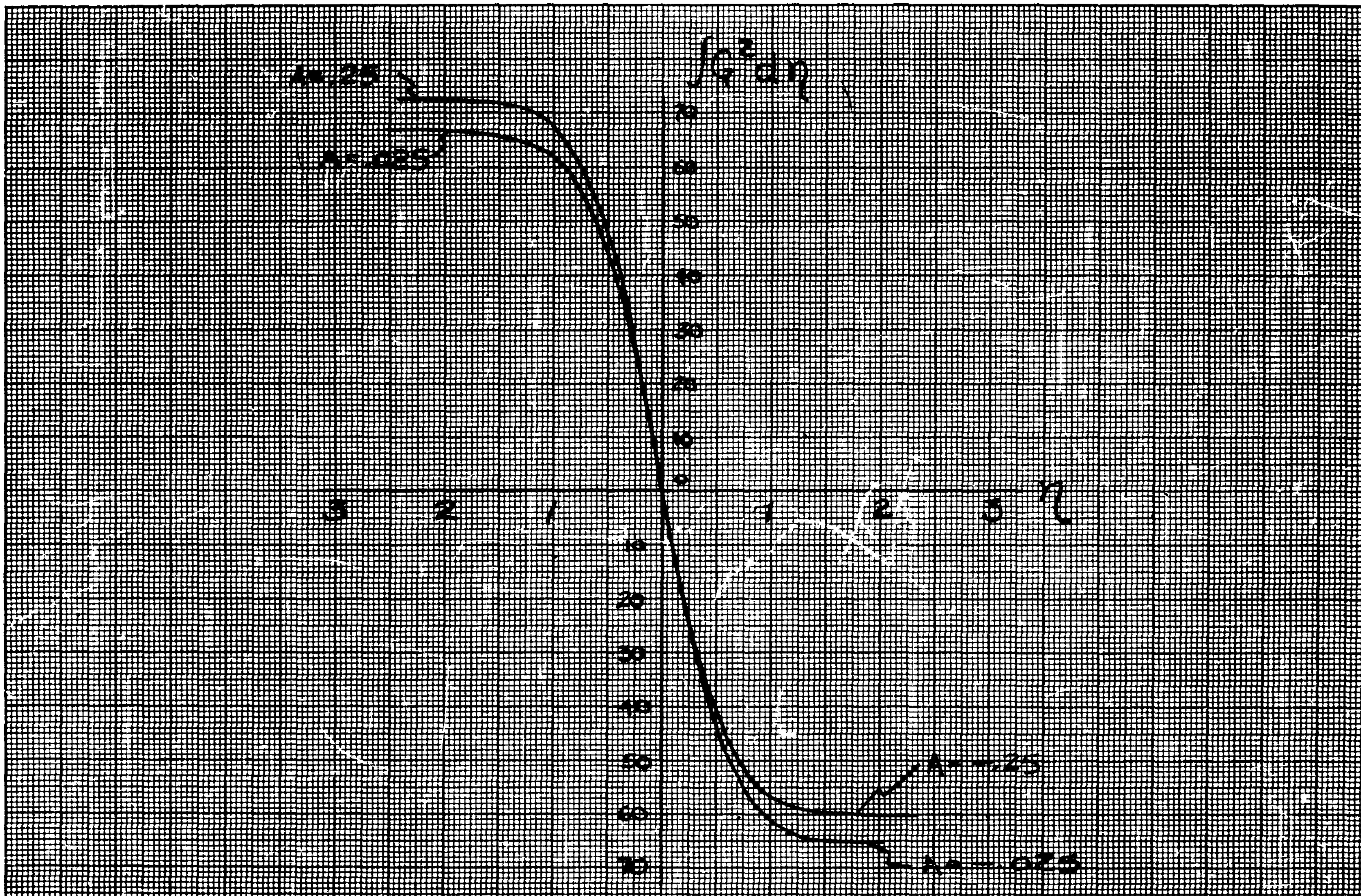


FIG. 7 STATIC PRESSURE ACROSS A CURVED JET

DISTRIBUTION

USACGSC	1
USAWC	1
USAPRDC	1
ARO, Durham	2
OCRD, DA	2
NATC	1
CRD, Physical Scn Div	1
USAERDL	2
USATAC	1
USATCDA	1
USATB	1
USATMC	2
USATSCH	1
USATRECOM	72
USATRECOM LO, USARDG (EUR)	2
USAEWES	2
TCLO, USAAVNS	1
Air Univ Lib	1
USNCEL	1
USASGCA	1
Canadian LO, USATSCH	3
BRAS, DAQMG (Mov & Tn)	4
USASG, UK	1
ASTIA	10
HumRRO	2
US Patent Ofc, Scn Lib	1
ODDR&E	1
Hum Engr Lab	1
MOCOM	3
USSTRICOM	1
AMC	6
BUWEPS, DN	3
CNO	2
CNR	25
ONR, Chicago	1
ONR, FPO NY	7
ONR, NY	1
ONR, San Francisco	1
ONR, Pasadena	1
SE Area Contr Adm, ONR	1
Nav Res Lab	6

BUSHP, DN	2
Dav Tay Mod Bas	3
CMC	1
MCDC	1
Maritime Adm	1
Hq NASA	1
TIA	1
Aerophysics Company	10

Army Transportation Research Command, Fort Eustis, Virginia
A SOLUTION OF THE TWO-DIMENSIONAL TURBULENT VISCOUS CURVED JET USING THE IBM 7090 COMPUTER, by Gabriel D. Boehler, Aerophysics Co., Report AR 61-03, TRECOM tech rept 63-14, March 1963. 44p. incl. illus, tables, Contract Nonr-2747(00)

Unclassified Report

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1. Air Cushion Vehicle
2. Contract Nonr-2747(00)

(over)

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(over)

This solution is a straightforward extension of the classical straight jet solution. Its only limitation is that it assumes similar velocity profiles.

The solution is obtained by reducing the Navier Stokes partial differential equations of the curved jet flow to a third-order total differential equation of the Hartree-Skan type. This equation is integrated numerically on the IBM 7090 computer using a Runge-Kutta subroutine. The boundary conditions are carefully established and discussed. The complete program for the numerical solution is included in the report. A duplicate IBM deck can be obtained from Aerophysics Company upon request. Typical numerical results, in the form of velocity and pressure distribution across the jet, are presented and discussed.

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